

# Linear Collisionless Landau Damping in Hilbert Space

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The equivalence between the Laplace transform [Landau L., *J. Phys. USSR* **10** (1946), 25] and Hermite transform [Zocco and Schekochihin, *Phys. Plasmas* **18**, 102309 (2011)] solutions of the linear collisionless Landau damping problem is proven.

**PACS codes:**

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## 1. Hermite revival

Vibrations in plasmas can be damped even in the absence of collisions. This phenomenon is known as Landau damping (Landau 1946). Landau damping acts on different types of waves: Langmuir waves, sound waves, kinetic Alfvén waves, drift waves, and many more. It basically occurs any time the momentum associated to a wave propagating in a plasma can sample regions of velocity-space where the plasma distribution function is prone to the formation of a singularity. A particularly interesting case is that of kinetic Alfvén waves (KAW) (Hasegawa & Chen 1975), and will be studied here. These waves are of pivotal importance in many physical phenomena in magnetised plasmas such as magnetic reconnection (Yamada *et al.* 2010), auroral electromagnetic turbulence (Louarn *et al.* 1994), and astrophysical gyrokinetics (Schekochihin *et al.* 2009), for instance.

A simple hybrid fluid-kinetic model that supports kinetic Alfvén waves *and* their Landau damping was introduced by Zocco & Schekochihin (2011). There, an efficient way to simulate the model equations numerically via the Hermite representation of velocity-space was proposed. This proved to be useful to study electron heating and nonlinear Landau damping of kinetic Alfvén waves in collisionless magnetic reconnection (Loureiro *et al.* 2013).

While some physical insight on nonlinear Landau damping can be obtained by “brute force” numerical simulations, (Loureiro *et al.* 2013), it is still unclear what is the relation between the original linear result of Landau (1946) and the Hermite representation of velocity-space (Hammett *et al.* 1993; Smith 1997; Sugama *et al.* 2001; Zocco & Schekochihin 2011). In this work we address this issue, and prove the equivalence of the two treatments.

## 2. Equations

We briefly review the system of equations studied. Details can be found in Zocco & Schekochihin (2011). We start with the collisionless electron drift-kinetic equation (Frieman & Chen 1982) for  $h_e = F_e - F_{0e}(1 + e\varphi/T_{0e})$

$$\frac{\partial h_e}{\partial t} + \mathbf{v}_E \cdot \nabla h_e + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla h_e = -\frac{eF_{0e}}{T_{0e}} \frac{\partial}{\partial t} \left( \varphi - \frac{v_{\parallel} A_{\parallel}}{c} \right), \quad (2.1)$$

where  $\mathbf{v}_E = cB_0^{-1}(-\partial_y \varphi \mathbf{e}_x + \partial_x \varphi \mathbf{e}_y)$  is the  $\mathbf{E} \times \mathbf{B}$  drift velocity,  $\hat{\mathbf{b}} \cdot \nabla = \partial_z - B_0^{-1} \{A_{\parallel}, \}$ ,  $\varphi$  and  $A_{\parallel}$  are the electrostatic and magnetic potential,  $\{, \}$  is the Poisson bracket, and

$$F_{0e} = \frac{n_{0e}}{[\pi v_{the}^2]^{3/2}} e^{-\frac{v_{\parallel}^2 + v_{\perp}^2}{v_{the}^2}} \quad (2.2)$$

is the Maxwellian equilibrium with temperature  $T_{0e} = m_e v_{the}^2 / 2$ . Equation (2.1) describes the statistical properties of a magnetised electron species for low-frequency anisotropic fluctuations in the presence of a mean magnetic field. Here, this is a constant straight magnetic field  $\mathbf{B}_0 = B_0 \hat{\mathbf{b}}$ , whose direction defines the z axis.

We introduce a formal mass ratio expansion for the electron perturbed distribution function, so that to zeroth order

$$h_e = \left( -\frac{e\varphi}{T_{0e}} + \frac{\delta n_e}{n_{0e}} + \frac{v_{\parallel} u_{\parallel e}}{T_{0e}} m_e \right) F_{0e} + g_e + \mathcal{O} \left( \frac{m_e}{m_i} \right), \quad (2.3)$$

here  $\delta n_e = \int d^3 \mathbf{v} h_e$ ,  $u_{\parallel e} = n_{0e}^{-1} \int d^3 \mathbf{v} v_{\parallel} h_e$ , and

$$\int d^3 \mathbf{v} (1, v_{\parallel}) g_e \equiv 0. \quad (2.4)$$

Using expression (2.3) in Eq. (2.1), and taking the zeroth moment we obtain the electron continuity equation

$$\frac{d}{dt} \frac{\delta n_e}{n_{0e}} = -\hat{\mathbf{b}} \cdot \nabla u_{\parallel e} \quad (2.5)$$

with  $d/dt = \partial_t + \mathbf{v}_E \cdot \nabla$ .

The first moment of Eq. (2.1) yields the generalized Ohm's law

$$\frac{d}{dt} (A_{\parallel} - d_e^2 \nabla_{\perp}^2 A_{\parallel}) = -c \frac{\partial \varphi}{\partial z} + \frac{T_{0e} c}{e} \hat{\mathbf{b}} \cdot \nabla \left[ \frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}} \right] + \frac{m_e c}{e} \frac{1}{n_{0e}} \frac{d}{dt} n_{0e} u_{\parallel i}, \quad (2.6)$$

with

$$\frac{\delta T_{\parallel e}}{T_{0e}} \equiv \frac{1}{n_{0e}} \int d^3 \mathbf{v} 2 \frac{v_{\parallel}^2}{v_{the}^2} g_e. \quad (2.7)$$

In Eq. (2.6), we used parallel Ampere's law

$$u_{\parallel e} = \frac{e}{m_e c} d_e^2 \nabla_{\perp}^2 A_{\parallel} + u_{\parallel i} \quad (2.8)$$

where  $d_e = c/\omega_{pe}$  is the electron skin depth, and  $\omega_{pe}$  the electron plasma frequency. An equation for  $g_e$  is derived after using Eqs. (2.5) and (2.6) in Eq. (2.1). The result is

$$\frac{dg_e}{dt} + v_{\parallel} \left[ \hat{\mathbf{b}} \cdot \nabla g_e - F_{0e} \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_{\parallel e}}{T_{0e}} \right] = F_{0e} \left( 1 - 2 \frac{v_{\parallel}^2}{v_{the}^2} \right) \hat{\mathbf{b}} \cdot \nabla \left[ \left( \frac{e}{m_e c} d_e^2 \nabla_{\perp}^2 A_{\parallel} + u_{\parallel i} \right) \right]. \quad (2.9)$$

The kinetic information is embedded in the function

$$\frac{\delta T_{\parallel e}}{T_{0e}} \equiv \frac{1}{n_{0e}} \int d^3 \mathbf{v} 2 \frac{v_{\parallel}^2}{v_{the}^2} g_e. \quad (2.10)$$

The system of equations is closed by solving for the ion dynamics, imposing quasineutrality (Zocco & Schekochihin 2011)

$$\frac{\delta n_e}{n_{0e}} = \frac{\delta n_i}{n_{0i}}, \quad (2.11)$$

and using the ion solution (Coppi *et al.* 1979; Antonsen & Coppi 1981; Crew *et al.* 1982; Pegoraro *et al.* 1989; Porcelli 1991; Zocco & Schekochihin 2011; Connor *et al.* 2012)

$$\frac{\delta n_i}{n_{0i}} = \int_{-\infty}^{+\infty} dp e^{ipx} F(p\rho_i) \frac{Ze\varphi}{T_{0i}} \equiv \hat{F} \frac{Ze\varphi_{\mathbf{k}}}{T_{0i}}, \quad (2.12)$$

with

$$F(p\rho_i) = -(1 - \Gamma_0), \quad (2.13)$$

$\tau = T_{0i}/T_{0e}$ ,  $Z$  is the charge number,  $k^2 = k_x^2 + k_y^2$ , and  $\Gamma_n = \exp[-p^2 \rho_i^2/2] I_n(p^2 \rho_i^2/2)$ , where  $I_n$  is the modified Bessel function (Abramowitz & Stegun 1972). The “hat” on  $F(p\rho_i)$  is a short-hand notation for the inverse Fourier transform. The ion solution guarantees that  $u_{\parallel i} = 0$ . In the truly collisionless case considered here, we can solve the electron kinetic equation (2.9). We linearize Eq. (2.9) using perturbations of the form  $g_e \propto \exp[-i\omega t + ik_{\parallel} v_{the}]$ .

After using standard algebra, we find (Zocco & Schekochihin 2011),

$$\frac{\delta T_{\parallel e}}{T_{0e}} = -\frac{k_{\parallel}}{|k_{\parallel}|} \frac{u_{\parallel e}}{v_{the}} \frac{Z(\zeta) - 2\zeta[1 + \zeta Z(\zeta)]}{1 + \zeta Z(\zeta)}, \quad (2.14)$$

where  $\zeta = \omega/(|k_{\parallel}| v_{the})$ , and

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dt \frac{e^{-t^2}}{t - \zeta} \quad (2.15)$$

is the plasma dispersion function (Fried *et al.* 1968).

We can replace Eq. (2.14) in Eq. (2.6) to obtain (Zocco & Schekochihin 2011)

$$\left[ \zeta^2 - \frac{\tau}{Z} \frac{k_{\perp}^2 d_e^2}{1 - \Gamma_0 (k_{\perp}^2 \rho_i^2/2)} \right] [1 + \zeta Z(\zeta)] = \frac{1}{2} k_{\perp}^2 d_e^2. \quad (2.16)$$

Looking for solutions with  $\zeta = \omega/(|k_{\parallel}| v_{the}) \ll 1$ , one gets the dispersion relation of shear and kinetic Alfvén wave

$$\omega_0 = \pm k_{\parallel} v_A k_{\perp} \rho_i \sqrt{\frac{1}{2} \left[ \frac{Z}{\tau} + \frac{1}{1 - \Gamma_0 (k_{\perp}^2 \rho_i^2/2)} \right]}, \quad (2.17)$$

where the damping rate is found by solving Eq. (2.16) perturbatively in  $\gamma/\omega_0 \ll 1$ , seeking for a solution  $\omega = \omega_0 + i\gamma$ :

$$\gamma = -|k_{\parallel}| v_A \frac{k_{\perp}^2 \rho_i^2}{4} \sqrt{\pi \frac{m_e}{m_i} \frac{Z^3}{\tau^2 \beta_e}}. \quad (2.18)$$

In the following, we solve analytically Eq. (2.9) by using Hermite polynomials as a basis in Hilbert space, and prove analytically that the Hilbert space solution converges to Eq. (2.14) when we take to infinity the number of Hermite moments kept in the system.

### 3. Hilbert Space

We introduce the Hermite inverse transform defined as

$$\hat{g}_e(v_{\parallel}) = \sum_{m=2}^{\infty} \frac{H_m(\hat{v}_{\parallel})}{\sqrt{2^m m!}} \hat{g}_m F_{0e}(\hat{v}_{\parallel}^2), \quad (3.1)$$

with coefficients

$$\hat{g}_m = \frac{1}{n_{0e}} \int_{-\infty}^{\infty} d\hat{v}_{\parallel} \frac{H_m(\hat{v}_{\parallel})}{\sqrt{2^m m!}} \hat{g}_e(v_{\parallel}), \quad (3.2)$$

where  $\hat{v}_{\parallel} = v_{\parallel}/v_{the}$ , and  $\hat{g}_e = 2v_{the}^{-2} \int dv_{\perp} v_{\perp} \exp[-v_{\perp}^2/v_{the}^2] g_e$ . The resulting electron kinetic equation is

$$\begin{aligned} \frac{d}{dt} \hat{g}_m + v_{the} \hat{\mathbf{b}} \cdot \nabla \left( \sqrt{\frac{m+1}{2}} \hat{g}_{m+1} + \sqrt{\frac{m}{2}} \hat{g}_{m-1} - \delta_{m,1} \hat{g}_2 \right) \\ = -\sqrt{2} \delta_{m,2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel e} \end{aligned} \quad (3.3)$$

Hence, for the first Hermite moments we obtain

$$\frac{d}{dt} \hat{g}_2 + \sqrt{\frac{3}{2}} v_{the} \hat{\mathbf{b}} \cdot \nabla \hat{g}_3 = -\sqrt{2} \hat{\mathbf{b}} \cdot \nabla u_{\parallel e}, \quad (3.4)$$

for  $m = 2$ , and

$$\frac{d}{dt} \hat{g}_m + v_{the} \hat{\mathbf{b}} \cdot \nabla \left( \sqrt{\frac{m+1}{2}} \hat{g}_{m+1} + \sqrt{\frac{m}{2}} \hat{g}_{m-1} \right) = 0, \quad (3.5)$$

for  $m \geq 3$ . The moments  $g_0$  and  $g_1$  do not appear in the summation in Eq. (3.1) as they must be set to zero in order to satisfy Eq. (2.4).

Equation (3.5) seems to suggest we should look for an iterative solution. However, it manifests the typical problem of a kinetic system, where low-order moments are coupled to high-order ones, therefore requiring to solve an infinite number of equations in order to know the full kinetic dynamics. We take advantage of the scaling of Hermite coefficients with the Hermite order, and virtually solve for an infinite number of equations in a very compact form. We then prove that our solution is equivalent to the solution found by using Landau contour integration.

### 4. Landau-Hermite equivalence

We find useful to consider the following limit

$$\frac{\hat{g}_m}{\hat{g}_{m-1}} \sim \frac{k_{\parallel} v_{the}}{\sqrt{m} \omega} \ll 1, \quad (4.1)$$

with

$$\frac{k_{\parallel} v_{the}}{\omega \sqrt{m}} \ll 1, \quad \text{and} \quad \frac{k_{\parallel} v_{the}}{\omega} \sim 1, \quad (4.2)$$

which is true for large  $m \gg 1$ .

If there is an  $N \gg 1$  for which  $\hat{g}_{N+1} \ll \hat{g}_N$  in the sense of Eq. (4.1), then the  $N$ th component of the kinetic equation is

$$\omega \hat{g}_N = k_{\parallel} v_{the} \sqrt{\frac{N}{2}} \hat{g}_{N-1}. \quad (4.3)$$

We then use this solution for  $\hat{g}_N$  in the equation for the  $N - 1$  component and solve for the  $N - 1$  component as a function of the  $N - 2$  component

$$\left[ \omega - k_{\parallel} v_{the} \frac{k_{\parallel} v_{the} N/2}{\omega} \right] \hat{g}_{N-1} = k_{\parallel} v_{the} \sqrt{\frac{N-1}{2}} \hat{g}_{N-2}. \quad (4.4)$$

After  $n$  iterations we have

$$\hat{g}_{N-n} = k_{\parallel} v_{the} \sqrt{\frac{N-n}{2}} \hat{g}_{N-(n+1)} \frac{1}{\omega - k_{\parallel} v_{the} \frac{k_{\parallel} v_{the} (N-n+1)/2}{\omega - k_{\parallel} v_{the} \frac{k_{\parallel} v_{the} (N-n+2)/2}{\omega - k_{\parallel} v_{the} \frac{k_{\parallel} v_{the} N/2}{\omega}}}}. \quad (4.5)$$

Equation (4.5) is a *finite* continued fraction that can be used to generate the *infinite* one which is the exact solution of the collisionless problem. Effects of finite collisionality within this formalism have been considered somewhere else (Sugama *et al.* 2001; Loureiro *et al.* 2013; Hatch *et al.* 2013; Parker & Dellar 2014; Zocco *et al.* 2014). Now, when  $N - n = 3$ , we are able to write  $\hat{g}_3$  in Eq. (3.4) explicitly as a function of all other  $\hat{g}_m$  up to  $\hat{g}_N$ , and therefore the electron temperature perturbation in terms of the continued fraction, which does not need to be truncated. The result is

$$\frac{v_{the}}{u_{\parallel e}} \frac{\delta T_{\parallel}}{T_{0e}} = \frac{k_{\parallel}}{|k_{\parallel}|} \frac{2}{\zeta} \frac{1}{1 - \frac{1}{\zeta^2} \frac{3/2}{1 - \frac{1}{\zeta^2} \frac{4/2}{1 - \frac{1}{\zeta^2} \frac{5/2}{\ddots}}}} \equiv \frac{k_{\parallel}}{|k_{\parallel}|} \frac{2}{\zeta} \frac{1}{\mathcal{D}^{(2)}}. \quad (4.6)$$

We now prove that Eq. (4.6) and (2.14) are the same.

We rewrite Eq. (4.6) in the following way

$$\frac{v_{the}}{u_{\parallel e}} \frac{\delta T_{\parallel}}{T_{0e}} = \frac{k_{\parallel}}{|k_{\parallel}|} \frac{1}{\zeta} \frac{1 - \frac{1}{\zeta^2} \frac{2/2}{\mathcal{D}^{(2)}} - 1}{-\frac{1}{2\zeta^2}}, \quad (4.7)$$

and notice that

$$\frac{1}{\zeta} \frac{1 - \frac{1}{\zeta^2} \frac{2/2}{\mathcal{D}^{(2)}} - 1}{-\frac{1}{2\zeta^2}} = \frac{\frac{1}{\zeta} - \frac{1}{\zeta \mathcal{D}^{(1)}}}{-\frac{1}{2\zeta^2 \mathcal{D}^{(1)}}}, \quad (4.8)$$

where

$$\mathcal{D}^{(1)} = \frac{1}{1 - \frac{1}{\zeta^2} \frac{2/2}{1 - \frac{1}{\zeta^2} \frac{3/2}{1 - \frac{1}{\zeta^2} \frac{4/2}{\ddots}}}}. \quad (4.9)$$

Similarly, we rewrite the RHS of Eq. (4.8) as

$$\frac{\frac{1}{\zeta} - \frac{1}{\zeta \mathcal{D}^{(1)}}}{-\frac{1}{2\zeta^2 \mathcal{D}^{(1)}}} = \frac{\frac{1}{\zeta} + 2\zeta \left[ 1 - \frac{1}{2\zeta^2 \mathcal{D}^{(1)}} - 1 \right]}{1 - \frac{1}{2\zeta^2 \mathcal{D}^{(1)}} - 1}, \quad (4.10)$$

which implies

$$\frac{v_{the}}{u_{\parallel e}} \frac{\delta T_{\parallel}}{T_{0e}} = \frac{k_{\parallel}}{|k_{\parallel}|} \frac{\frac{1}{\zeta \mathcal{D}^{(0)}} + 2\zeta \left[ 1 - \frac{1}{\mathcal{D}^{(0)}} \right]}{1 - \frac{1}{\mathcal{D}^{(0)}}}, \quad (4.11)$$

with

$$\mathcal{D}^{(0)} = \frac{1}{1 - \frac{1}{\zeta^2} \frac{1/2}{1 - \frac{1}{\zeta^2} \frac{2/2}{1 - \frac{1}{\zeta^2} \frac{3/2}{\ddots}}}}. \quad (4.12)$$

Since

$$\frac{dZ(\zeta)}{d\zeta} = -2 [1 + \zeta Z(\zeta)], \quad (4.13)$$

with  $Z(0) = i\sqrt{\pi}$ , successive differentiation of Eq. (4.13) yields (McCabe 1984)

$$\frac{Z^{(n)}}{Z^{(n-1)}} = \frac{-2n}{2\zeta + Z^{(n+1)}/Z^{(n)}}, \quad (4.14)$$

and therefore [see also Eq. 7.1.15 of Abramowitz & Stegun (1972)]

$$Z(\zeta) = -\frac{1}{\zeta \mathcal{D}^{(0)}}. \quad (4.15)$$

Hence, we showed that

$$\frac{2}{\zeta} \frac{1}{\mathcal{D}^{(2)}} = -\frac{Z(\zeta) - 2\zeta [1 + \zeta Z(\zeta)]}{1 + \zeta Z(\zeta)}. \quad (4.16)$$

This concludes our proof.

## 5. Conclusion

We solved the problem of linear collisionless Landau damping for kinetic and shear Alfvén waves by using both the traditional Laplace transform approach and a Hermite transform.

We introduced a recursive formula for the coefficients of the inverse Hermite transform that allowed us to construct a finite continued fraction whose extension to infinite elements gave a new exact solution for the electron distribution function. We proved that this new solution is equivalent to the solution found by using Landau contour integration.

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